**[O2.17]**

# Generic implementation of meshless local strong form method


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## Introduction (300 words)

The simplicity, generality and efficiency are probably the main attractiveness of the meshless methods that have been under intense developed in recent years. This paper considers one of those methods, namely a relatively simple Meshless Local Strong form Method (MLSM) [1] that generalises several other mesh based and meshless strong form methods, e.g. Finite Difference Method, Local Radial Basis Functions Method, Finite Point Method, Diffuse Approximate Method, etc. MLSM is based on a Weighted Least Squares (WLS) approximants constructed over small local neighbourhood of the considered node. One of the most attractive features of MLSM is its dependency solely on the nodal positions, which enables a rather general formulation that can be directly implemented in a programming language with support for generic abstractions, such as C++11. In another words, the MLSM implementation is not dependent on dimensionality of the domain, on the approximation type, basis pool size, basis types, support size, and nodal positions. In this paper the generic implementation of MLSM is demonstrated by solving various problems in 1D, 2D and 3D on different domain shapes. It is shown that this implementation has little to no execution overhead over e.g. classical Finite Difference Method implementation despite its significant abstraction.

## Methods (300 words)

To obtain a solution of a PDE, it is first split on a spatial and temporal part. Temporal discretization is treated separately using standard methods, while the spatial part is solved using MLSM. The spatial part is assumed to be an elliptic boundary value problem $Lu = f$ on domain $\Omega$ with Dirichlet boundary conditions $u = u_0$ on $\partial\Omega$. First, $\Omega$ is discretized by choosing $N_I$ points in domain interior and $N_B$ points on the boundary.

The core of the spatial discretization is based on WLS approximation, constructed over local neighbourhood. Choose a point $p$ and $n$ of its neighbours, denoted by $s_1, \dots, s_n$. An approximation $\hat{u}$ of function $u$ is introduced in form $\hat{u}(x) = \sum_{i=1}^{m} \alpha_i b_i(x) = \boldsymbol{b}^T(x)\boldsymbol{\alpha}$.

where $b_i$ are arbitrary basis functions defined over the neighbourhood of $p$ and $\alpha_i$ are unknown coefficients. To obtain these coefficients, a weighted discrete $L_2$ norm

$$R^2 = \sum_{i=1}^{n} w_i \left( u(s_i) - \sum_{j=1}^{m} \alpha_j b_j(s_i) \right)^2,$$

is minimized, where positive numbers $w_i$ represents the weights associated with support nodes $s_i$. Above equation can be rearranged into minimization of $\|WB\boldsymbol{\alpha} - W\boldsymbol{u}\|_2^2$, where $B_{ij} = b_j(s_i)$ is a $n \times m$ matrix, $\boldsymbol{u}$ is the vector of function values, $u_i = u(s_i)$, and $W$ is a diagonal matrix of square roots of weights, $W_{ii} = \sqrt{w_i}$. This problem can be in general solved by computing the Moore-Penrose pseudoinverse of $B$, denoted by $B^+$ and expressing $\boldsymbol{\alpha} = (WB)^+ W\boldsymbol{u}$. Vector $\boldsymbol{\alpha}$ can be substituted back into approximation to obtain $\hat{u}(p) = \boldsymbol{b}^T(p)(WB)^+ W\boldsymbol{u}$.

Additionally, a linear partial differential operator $L$ can be applied to $\hat{u}$ simply by computing $(L\hat{u})(p) = (L\boldsymbol{b})^T(p)(WB)^+ W\boldsymbol{u} = \chi^L(p)\boldsymbol{u}$.

where $\chi^L(p) = (Lb)^T(p)(WB)^+W$ is called a shape function for operator $L$ in point $p$ and gives the discrete approximation of $L$ applied at $p$.

To solve given equation $Lu = f$ on $\Omega$ we approximate it with a linear equation $\chi^L(p)\boldsymbol{u} = f(p)$ for every discretization point $p$ in the interior of the domain. Accompanied by equations $u(p) = u_0(p)$ for boundary nodes, this now yields a sparse system of linear equations with $N_I n + N_B$ nonzero entries, which can be solved to obtain an approximation for $u$.

**Results (300 words)**

First the method is tested on 1D boundary value problem $u''(x) = \sin x, u(0) = 1, u'(0) = 1$.

Results are compared against an analytical solution in terms of accuracy with respect to the number of discretization nodes. It is demonstrated that MLSM has the same order of convergence as FDM and no implementation overhead. Convergence is also analysed regarding different precision floating point arithmetic. To solve a two or three dimensional problem, only very small modifications are necessary, which is demonstrated by solving $\nabla^2 u = 1$ and $u|_{\partial\Omega} = 0$ in two and three dimensions. Again, convergence against analytical solution is analysed for different setups up to $10^6$ nodes and the observed order of convergence matches the one obtained with FDM. The solution procedure is then generalized to arbitrary two and three dimensional shapes.

Next, we consider a classical cantilever beam problem from linear elasticity [2], where displacements of a beam, bent in one end by a parabolically distributed force $P$, are sought. A numerical solution is obtained by solving the Cauchy-Navier equations $(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u} = \vec{0}$. Again; convergence in $L_\infty$ norm in displacements and stresses towards an analytical solution up to $10^6$ nodes is shown. In addition, a drilled cantilever beam domain is also solved to demonstrate flexibility of MLSM regarding nodal positions.

Finally, a solution of the classical lid driven cavity problem [3] from computational fluid dynamics is considered up to $Re = 3200$ and up to $5\ 10^4$ nodes. This time, no analytical solution is known, but the problem is well studied and can be compared to already known results from literature [3, 4].

**Conclusions and Contributions (300 words)**

The paper describes a general strong form meshless method which generalizes many existing meshless methods, such as FDM, LRBFCM, FPM etc., along with a generic implementation of the method written in C++11. It is demonstrated that the method is suitable for solving energy transport problems, linear elasticity and fluid flow problems by solving benchmark cases in all three fields along with full convergence analyses. Furthermore, it is shown that the method generalizes well to different shapes and higher dimensions with little to no modifications.

It is also demonstrated that it is possible to have a generic implementation in a modern language using high level concepts that enables direct mapping of the mathematical description to the code without significant overhead.

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