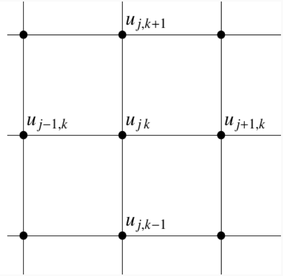
High order RBF-FD approximations with application to a scattering problem

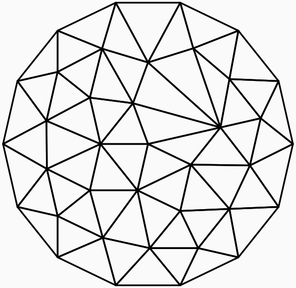
Jure Slak, Blaž Stojanovič, Gregor Kosec

"Jožef Stefan" Institute, Parallel and Distributed Systems Laboratory

19. 6. 2019, SpliTech 2019

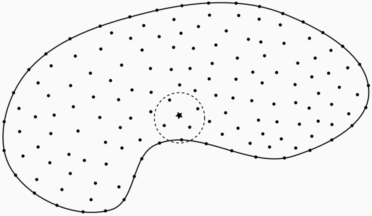
- Classical approaches:
  Finite Difference Method, Finite Element Method





- Problems: inflexible geometry, mesh generation
- Response: mesh-free methods (EFG, MLPG, FPM)

Domain discretization:



- Points x_i on the boundary and in the interior
- Point neighborhoods $N(x_i)$

Classical Finite Differences:

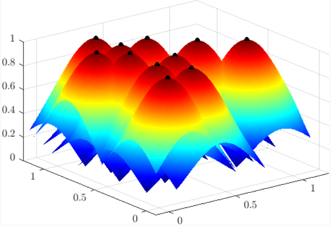
$$u''(x_i) \approx \frac{1}{h^2}u(x_{i-1}) - \frac{2}{h^2}u(x_i) + \frac{1}{h^2}u(x_{i+1})$$

Generalized Finite Differences:

$$(\mathcal{L}u)(x_i) \approx \sum_{x_j \in N(x_i)} w_j^i u(x_j)$$

+ exactness for a certain set of functions (e.g. monomials)

Exactness is imposed for Radial Basis Functions



- Given nodes $X = \{x_1, \ldots, x_n\}$ and a radial function $\varphi = \varphi(r)$
- Generate $\{\varphi_i := \varphi(\|\cdot - x_i\|), x_i \in X\}$

Imposing exactness of

$$(\mathcal{L}u)(x_i) \approx \sum_{x_j \in N(x_i)} w^i_j u(x_j)$$

for each φ_j for $x_j \in N(x_i)$, we get

$$\begin{bmatrix} \varphi(\|x_{j_1} - x_{j_1}\|) & \cdots & \varphi(\|x_{j_{n_i}} - x_{j_1}\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|x_{j_1} - x_{j_{n_i}}\|) & \cdots & \varphi(\|x_{j_{n_i}} - x_{j_{n_i}}\|) \end{bmatrix} \begin{bmatrix} w^i_{j_1} \\ \vdots \\ w^i_{j_{n_i}} \end{bmatrix} = \begin{bmatrix} (\mathcal{L}\varphi_{j_1})(x_i) \\ \vdots \\ (\mathcal{L}\varphi_{j_{n_i}})(x_i) \end{bmatrix}$$

RBF-FD + Monomial augmentation

Enforce consistency up to certain order, e.g. for constants

$$\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_\varphi \\ 0 \end{bmatrix}$$

In general:

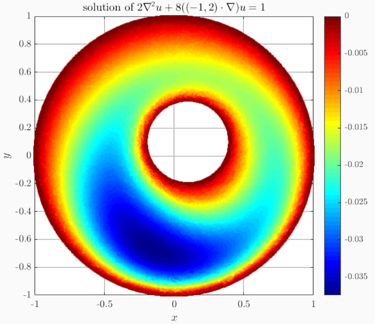
$$\begin{bmatrix} A & P \\ P^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_\varphi \\ \boldsymbol{\ell}_p \end{bmatrix},$$

where

$$P = \begin{bmatrix} p_1(\boldsymbol{x}_1) & \cdots & p_s(\boldsymbol{x}_1) \\ \vdots & \ddots & \vdots \\ p_1(\boldsymbol{x}_n) & \cdots & p_s(\boldsymbol{x}_n) \end{bmatrix}, \quad \boldsymbol{\ell}_p = \begin{bmatrix} (\mathcal{L}p_1)|_{\boldsymbol{x}=\boldsymbol{x}^*} \\ \vdots \\ (\mathcal{L}p_s)|_{\boldsymbol{x}=\boldsymbol{x}^*} \end{bmatrix}.$$

Problem:

$$
\begin{aligned}
\mathcal{L}u &= f \quad \text{on } \Omega, \\
u &= u_0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

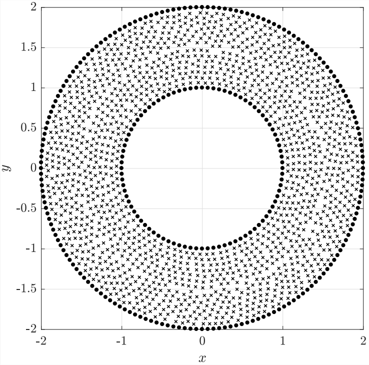


1. Discretize domain Ω
2. Find neighborhoods $N(x_i)$
3. Compute weights $\boldsymbol{w}^i$ for approximation of $\mathcal{L}$ over $N(x_i)$
4. Assemble weights in a sparse system $Wu = f$
5. Solve the sparse system $Wu = f$
6. Approximate/interpolate the solution

Test case: Poisson problem

$$\mathcal{L}u = f \quad \text{on } \Omega,$$
$$u = u_0 \quad \text{on } \partial\Omega,$$

Domain Ω is an annulus.
PHS are used: $\varphi(r) = r^3$.
Augmentation up to order m.
Maximally 45 monomials.
65 closest neighbours.

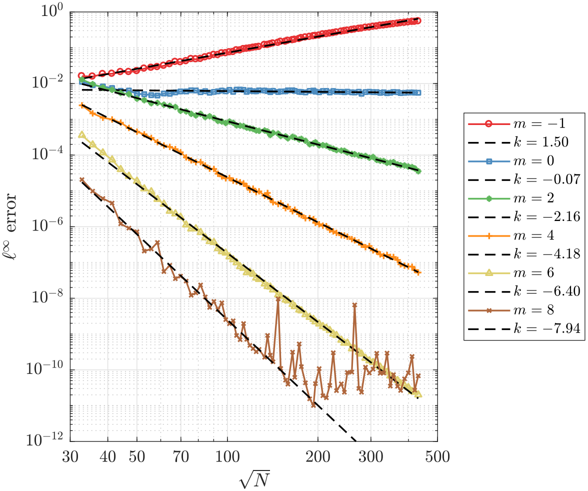


Test case: Poisson problem

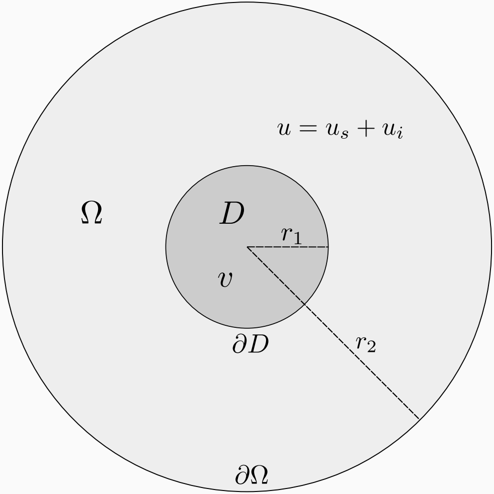
JSI

convergence orders match predictions

ℓ_1 and ℓ_2 errors similar



Anisotropic cylindrical scatterer. Let v be the (complex-valued) field inside the scatterer and $u = u^s + u^i$ outside.



Test case: Scattering problem

Discretized model:

$$\nabla \cdot A_\mu \nabla v + \epsilon_r k^2 \, v = 0 \qquad \text{in} \quad D,$$

$$\nabla^2 u^s + k^2 \, u^s = 0 \qquad \text{in} \quad \Omega \setminus D,$$

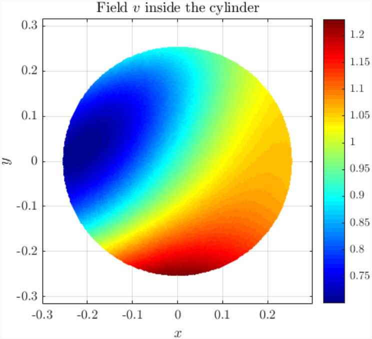
with boundary conditions

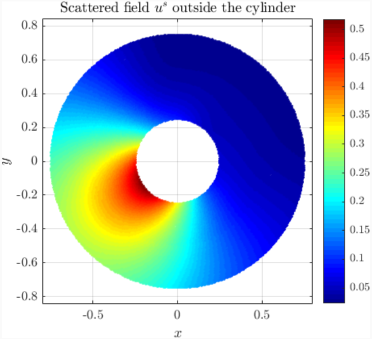
$$v - u^s = u^i \qquad \text{on} \quad \partial D,$$

$$\frac{\partial v}{\partial \vec{n}_{A_\mu}} - \frac{\partial u^s}{\partial \vec{n}} = \frac{\partial u^i}{\partial \vec{n}} \qquad \text{on} \quad \partial D,$$

$$\frac{\partial u^s}{\partial \vec{n}} + \left(ik + \frac{1}{2r_2} \right) u^s = 0 \qquad \text{on} \quad \partial \Omega.$$

Around 90 000 nodes, augmentation of order 4.

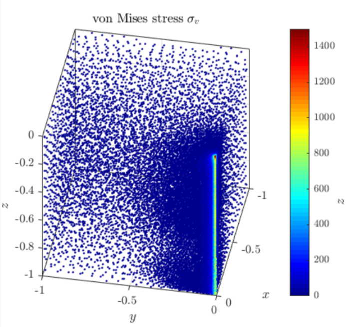


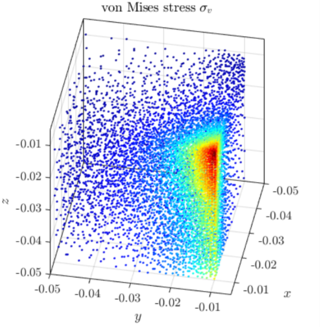


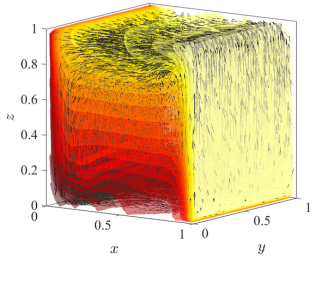
Magnitude of v.

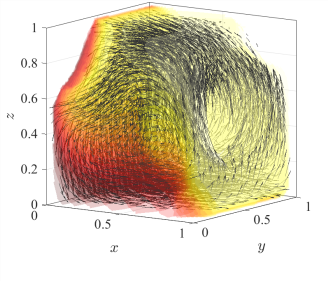
Magnitude of u^s.

Additional examples

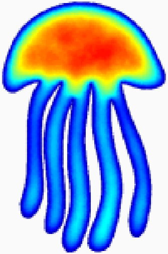








All computations were done using open source Medusa library.



Medusa

Coordinate Free Mehless Method implementation

`http://e6.ijs.si/medusa/`

Thank you for your attention!

Acknowledgments: FWO Lead Agency project: G018916N Multi-analysis of fretting fatigue using physical and virtual experiments, the ARRS research core funding No. P2-0095 and Young Researcher program PR-08346.